

**IZHANOVA KAMILA ALIBEKOVNA**

**BOUNDARY VALUE PROBLEMS FOR A TWO-DIMENSIONAL LOADED  
HEAT EQUATION**

**ABSTRACT**

**of the dissertation for the degree of Doctor of Philosophy (PhD) in the  
educational program 8D05401-Mathematics**

**Relevance of the topic.** In modern mathematics and applied science there is a growing interest on models describing processes with memory, heredity, fractal structures, and large-scale inhomogeneities. Such models often cannot be sufficiently described using classical integer-order differential equations. Therefore, fractional-order differential equations have gained particular popularity in recent decades, providing a more flexible tool for modeling complex systems, in which the nonlinearity, delay, and nonstationarity are present.

Mathematical models of nonlocal physical and biological fractal processes are also based on loaded differential equations with fractional partial derivatives., A.M.Nakhushev provided a detailed bibliography on loaded equations, including various applications of loaded equations as a method for studying problems in mathematical biology, mathematical physics, mathematical modeling of nonlocal processes, and continuum mechanics with memory. Fractional diffusion-wave equations are widely used in various fields of science and engineering. E. Shitikova presents a review of recent research on the application of fractional calculus to linear viscoelasticity models used in problems of dynamic solid mechanics.

Various methods exist for solving these equations: the Green's function method, the finite sine transform method, the separation of variables method, the finite difference method, the Adomian expansion method, and the finite difference scheme. Recently, Kasemi and Rostami proposed a new local discontinuous Galerkin method for the fractional-time diffusion wave equation. The equation was solved by defining a discontinuous finite element of degree one with interpolated coefficients.

From a mathematical perspective, it is of interest to study boundary value problems for the heat equation with a fractional load, where the loaded term is considered as a fractional derivative or a fractional integral. Boundary value problems for loaded differential operators can be interpreted as a functional-differential operator of the form:  $Lu + \gamma Mu = f$ , where  $L$  is the differential term and  $M$  is the loaded term. The loaded term in the equation is considered a perturbation. Moreover, it contains a trace of some operations from the sought-after

solution  $u$ . If the loaded term contains a trace of the fractional derivative of the sought-after function, then we are dealing with nonlocality, that is, the current state of the system depends on its history, but this dependence is expressed through an integral over the past values of the function. Some studies have shown that the existence and uniqueness of solutions to fractionally loaded boundary value problems in certain functional classes depend on the order of the fractional derivative in the loaded term. It has also been shown that for a boundary value problem with a load on a spatial variable, for certain values of the order of the fractional derivative, the spectral case happens; namely, the resulting Volterra integral equation contains an eigenfunction. Similar Volterra integral equations also arise in the study of boundary value problems in degenerate domains.

**Purpose of research.** The formulation and solution of boundary value problems for a loaded equation is the formulation and solution of boundary value problems for loaded equations containing a fractional integro-differentiation operator; establishing conditions for the existence and uniqueness of their solutions.

**Research objectives:**

1. Formulate boundary value problems (seven in total) for loaded equations and/or fractional-order equations, considering various types of fractional operators (Riemann-Liouville, Caputo), spatial dimensions, and domain geometries (including non-cylindrical domains).

2. Reduce the problems to integral or differential equations and obtain exact solutions for some of the resulting integral or integro-differential equations, representing them in closed form using special functions of fractional calculus.

3. For other posed problems, estimate the kernel of the resulting integral equations based on the asymptotic behavior of the special functions.

4. Conduct a qualitative analysis of the problems, classifying loaded terms as weak or strong perturbations depending on their influence on the uniqueness of the solution, identify and analyze the critical "spectral case" in which the problem has a non-unique solution, and find the corresponding eigenfunction of the homogeneous problem.

5. Formulate and prove theorems on the existence and uniqueness of solutions to boundary value problems under certain conditions on the initial data and parameters of the loaded term.

**Research methods.** The work uses methods of the general theory of differential equations, functional analysis and fractional calculus, methods of integral transformations and the fundamentals of the theory of singular integral equations.

**Scientific novelty.** Previously, boundary value problems for a fractionally loaded one-dimensional heat conduction equation were studied in the case

where the loaded term of the equation is represented as a fractional derivative of an order smaller than the order of the equation. Now, new types of boundary value problems for partial differential equations of fractional order have been posed and studied, in which the loaded terms include: a fractional integral, fractional derivatives with respect to both the time and spatial variables, and a product with a power coefficient. The loading points move both along a curve (in the one-dimensional case) and along a surface (in the two-dimensional, multidimensional case). It has been established that the loaded term can act as: a weak perturbation preserving the uniqueness of the solution (typical for fractional integrals or low-order derivatives); a strong perturbation leading to non-uniqueness and the existence of nontrivial solutions of the homogeneous problem (typical for high-order fractional derivatives). Furthermore, a "spectral case" was discovered for problems with a power coefficient in the loaded term. It was proven that for an exponent  $\mu$  equal to the difference in the orders of the derivatives in the equation and the loaded term ( $\mu = \beta - \alpha$ ), the homogeneous problem acquires a spectrum and has an infinite set of nontrivial solutions.

#### **Theoretical and practical significance of the research.**

This work is mainly theoretical. It contributes to the modern theory of boundary value problems for partial differential equations (PDEs) by considering a class of equations with two distinct characteristics: a fractional operator in the principal part of the equation and/or in the loaded term of the problem equation. It goes beyond classical PDE theory and addresses nonlocal problems, where the behavior of the equation at a point depends on the behavior of the solution over the entire domain or at specific points/instants of time. Therefore, the theoretical value of this work lies in the development of the theory of nonlocal problems. A key theoretical discovery is the establishment of a direct connection between the type of loaded term (e.g., fractional derivative or fractional integral) and the fundamental property of uniqueness of the solution. This provides a key criterion for classifying such problems and investigating their well-posedness. The results are relevant to various fields, including fractional viscoelasticity, thermodynamics of media with memory, population dynamics with nonlocal interactions, and financial mathematics with heredity effects.

#### **Main results presented for defense:**

1. Statement of boundary value problems for differential equations with a fractional operator in the principal part and/or a loaded term in the form of a fractional derivative or integral as a separate class of nonlocal problems whose properties cannot be reduced to those of classical partial differential equations.
2. Reduction of the stated boundary value problems to integral or differential equations and their study.

3. Constructing solutions to integral equations in explicit form and/or estimating their kernels based on the asymptotic behavior of special functions.

4. Solvability of the stated boundary value problems depending on the specific type of the loaded term.

5. Finding an eigenfunction for a problem with a power coefficient in the loaded term.

#### **Reliability and validity of the results.**

The constructive nature of the methods used ensures the reliability and validity of the study. General conclusions are formulated as theorems, and their proofs are provided.

**Publications.** 13 works have been published on the topic of the dissertation, of which 4 articles were published in journals included in the Scopus, and 9 works were published in collections of proceedings of international scientific conferences.

#### **Approval of the work.**

Based on the results of her dissertation, presentations were made at international conferences and conferences abroad:

1. Abstracts of the VII World Congress of Turkic World Mathematicians (Turkestan, September 20-23, 2023).

2. The VII International Scientific Conference "Nonlocal Boundary Value Problems and Related Problems of Mathematical Biology, Informatics and Physics" (B&NAK-2023). (Nalchik, December 4-8 2023).

3. Evolution Equations, Approximation and Spectral Optimization: book of abstracts of International Summer School & Conference (Almaty, September 11-14, 2024).

4. Actual problems of applied mathematics and information technologies Al-Khwarizmi 2024: abstracts of the IX international scientific conference, dedicated to the 630th anniversary of the birth of Mirzo Ulugbek (Tashkent, October 22-23, 2024).

5. Annual International April Mathematical Conference (Almaty, April 1-4 2025).

6. Artificial Intelligence and Inverse Problems in Science, Technology and Industry: Proceedings (Astana, April 14-16 2025).

7. 15-th ISAAC Congress: Abstract book (Astana, July 21-25, 2025).

8. The 10th International Conference on Differential and Functional Differential Equations dedicated to the memory of academician S.P. Novikov: Abstracts (Moscow, August 17-24, 2025).

9. "Current issues of interdisciplinary scientific research" organized on the occasion of the 100th anniversary of Academician E.A. Buketov (Karaganda, June 17-20 2025).

**The doctoral candidate's contribution to the preparation of each publication.** In 13 works completed jointly with academic supervisors and co-authors, the academic supervisors set the task, while the doctoral student independently formulated the main and additional results and presented evidence for them.

**Structure and volume of the dissertation.** The dissertation, which is 74 pages long, consists of the following structural elements: introduction, two sections, conclusion, and list of sources used.

**Number of sources used - 64.**

**Keywords.** Loaded equation, heat equation, diffusion equation, two-dimensional loaded heat equation.